Contractivity theorems in real ordered Banach spaces with applications to relative operator bounds, ergodic projections and conditional expectations

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Abstract

This paper provides various "contractivity" results for linear operators of the form $I-C$ where C are positive contractions on real ordered Banach spaces X. If A generates a positive contraction semigroup in Lebesgue spaces $L^p(\mu)$, we show (M. Pierre's result) that $A(\lambda-A)^{-1}$ is a *"contraction on the positive cone"*, i.e. $\left\|A(\lambda-A)^{-1}x\right\| \leq \|x\|$ for all $x \in L^p_+(\mu)$ $(\lambda > 0)$, provided that $p \geqslant 2$. We show also that this result is not true for $1 \leq p < 2$. We give an extension of M. Pierre's result to general ordered Banach spaces X under a suitable *uniform monotony* assumption on the duality map on the positive cone X_+. We deduce from this result that, in such spaces, $I-C$ is a contraction on X_+ for any positive *projection* C with norm 1. We give also a *direct* proof (by E. Ricard) of this last result if additionally the norm is smooth on the positive cone. For any positive contraction C on *base-norm spaces* X (e.g. in real $L^1(\mu)$ spaces or in preduals of hermitian part of von Neumann algebras), we show that $N(u-Cu) \leq \|u\|$ $\forall u \in X$ where N is the canonical half-norm in X. For any positive contraction C on *order-unit spaces* X (e.g. on the hermitian part of a C^* algebra), we show that $I-C$ is a contraction on X_+. Applications to relative operator bounds, ergodic projections and conditional expectations are given.

1 Introduction

In this paper, we give various "contractivity" results for perturbations of the identity by linear positive contractions in real ordered Banach spaces. This work was motivated initially by the computation of relative operator bounds in a context of perturbation of generators of *positive semigroups of contractions* (in connection e.g. with [5][1] and of interest e.g. for perturbation theory of submarkovian operators with Levy-type structure steeming from Dirichlet forms [7]).

It is well-known (see e.g. [24]) that if we consider a semibounded (say bounded from above) self-adjoint operator

$$A : D(A) \subset H \to H$$

in a Hilbert space H and if

$$S : D(A) \to H$$

is A-bounded then the *relative A-bound* of S (i.e. the infimum of constants $a > 0$ for which there exists a constant $b > 0$ such that

$$\|Sx\| \leq a \|Ax\| + b \|x\|, \ x \in D(A))$$

is equal to

$$\lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(H)}.$$

The initial motivation of this work was to look for conditions under which this property holds true in a Hilbert space when A is no longer self-adjoint or, more generally, when A is a generator of a C_0-semigroup $(U(t))_{t \geqslant 0}$ on a Banach space. This question turned out to have some connection with *"contractivity"* properties of perturbations of the identity by linear contractions in (ordered) Banach spaces of own interest which in turn have useful applications.

We show first that the property above remains true in a Hilbert space setting provided there exists some $\overline{\lambda} \in \mathbb{R}$ such that $A + \overline{\lambda}$ generates a contraction semigroup (see Corollary 3). More generally, in the context of generators A of C_0-semigroups $(U(t))_{t \geqslant 0}$ on Banach spaces X, the relative A-bound of S coincides with $\lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)}$ provided that

$$\limsup_{\lambda \to +\infty} \left\| A(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \leq 1.$$

On the other hand, it is easy to see that

$$\limsup_{\lambda \to +\infty} \left\| A(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \leq \limsup_{t \to 0} \left\| U(t) - I \right\|_{\mathcal{L}(X)} \tag{1}$$

and this leads us naturally to wonder if $A(\lambda - A)^{-1}$ or $U(t) - I$ could be contractions when A generates a C_0-semigroup of contractions $(U(t))_{t \geqslant 0}$. (Note that

$$A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I$$

and $\lambda(\lambda - A)^{-1}$ is a contraction.) It turns out that this cannot be true in general. Indeed, in an earlier version of this paper, we proved general *negative results* for submarkovian C_0-semigroups in Lebesgue spaces $L^p(\Omega, \mu)$, over a metric space Ω, for large p's (and also for p close to 1 in the symmetric case) provided that a suitable smoothing effect on $L^\infty(\Omega, \mu)$ is assumed. In particular, for stochastic (i.e. norm-preserving on the positive cone) positive C_0-semigroups on $L^1(\Omega, \mu)$ with a suitable "dual smoothing effect", we have the sharp results

$$\limsup_{\lambda \to +\infty} \left\| A(\lambda - A)^{-1} \right\|_{\mathcal{L}(L^1)} = \left\| U(t) - I \right\|_{\mathcal{L}(L^1)} = 2 \ \ (\forall t > 0). \tag{2}$$

These general counter-examples suggest fully that a priori we cannot expect $A(\lambda - A)^{-1}$ and $I - U(t)$ to be contractions outside the realm of Hilbert spaces. Despite this fact, we show in this paper how suitable "contractivity" properties *still hold* in suitable (non hilbertian) ordered Banach spaces. Indeed, this paper is devoted to various "contractivity" results for operators of the form

$$I - C$$

where C denotes linear positive contractions in suitable ordered Banach spaces. This study was triggered by the unsuspected fact that if A generates a positive semigroup (i.e. leaving invariant the positive cone X_+) on a real ordered Banach space with a Riesz norm and if $S : D(A) \cap X \to X$ is positive, i.e.

$$S : D(A) \cap X_+ \to X_+,$$

then the relative A-bound of S is equal to

$$\lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \tag{3}$$

(as in Hilbert space settings) provided that $A(\lambda - A)^{-1}$ is a *"contraction on the positive cone"* in the sense

$$\sup_{\|x\| \leq 1, \ x \in X_+} \left\| A(\lambda - A)^{-1} x \right\|_X \leq 1 \ \ (\lambda > 0).$$

This explains why "contractions on the positive cone" (which are *not* contractions on the whole space) are worth studying. This paper provides various theorems in this direction with applications to relative operator bounds, ergodic projections and conditional expectations (in both classical or noncommutative contexts). As far as we know, these results appear here for the first time. Three classes of ordered Banach spaces (and their duals) are involved in this study:

(i) Base-norm ordered Banach spaces (see the definition in Section 3). Such spaces cover e.g. real $L^1(\mu)$ spaces, the space of real Borel measures on a metric space endowed with the total variation norm, the duals of the hermitian part of C^* algebras or the preduals of the hermitian part of von Neumann algebras.

(ii) Order-unit ordered Banach spaces (see the definition in Section 3). Such spaces cover e.g. real $L^\infty(\mu)$ spaces, real $C(K)$ spaces or more generally (hermitian parts of) C^* algebras. (We point out that there is a perfect symmetry (*duality*) between base-norm spaces and order-unit spaces, see [15].)

(iii) The class of ordered Banach spaces whose duality map is uniformly monotone on the positive cone X_+ in the sense (6) below. This class covers the Lebesgue spaces $L^p(\mu)$ with $p \geqslant 2$ (and also noncommutative L^p spaces with $p \geqslant 2$, see [20]).

For reader's convenience, Section 3 is devoted to reminders on some basic definitions and results on real ordered Banach spaces we need in this paper.

Our main results are:

We show first that a bounded operator $B \in \mathcal{L}(X)$ is a contraction on the positive cone, in the sense

$$\sup_{\|x\|\leq 1,\ x\in X_+} \|Bx\|_X \leq 1,$$

if and only if the dual operator B' satisfies a sublinear contraction

$$N'\left(B'x'\right) \leq \left\|x'\right\|_{X'} \quad \forall x' \in X'$$

where N' is the canonical half-norm (see the definition in Section 3) in X'; we show also that

$$N\left(Bx\right) \leq \|x\|_X \quad \forall x \in X$$

(N is the canonical half-norm in X) if and only if B' is a contraction on the dual positive cone X'_+ (see Lemma 6). Notice that in a Banach lattice or on (hermitian elements of) a C^* algebra we have

$$N(x) = \|x_+\|\,,\ x \in X$$

where x_+ is the positive component of x (see e.g. [2]).

It turns out that in Lebesgue spaces $L^p(\mu)$ with

$$p \geqslant 2,$$

$A(\lambda - A)^{-1}$ is a contraction on the positive cone $L^p_+(\mu)$ for any generator A of a positive contraction C_0-semigroup in $L^p(\mu)$ (see Theorem 12). This unsuspected result is due to M. Pierre (private communication). This result is false once $p < 2$ (see a counter-example by E. Ricard in Remark 27; see also Theorem 15 and Remark 17 below for general counter-examples for submarkovian semigroups if p is *close* to 1 and Remark 20(ii)).

It follows from Theorem 12 and Lemma 6 that

$$\left\|(A(\lambda - A)^{-1} f)^{\pm}\right\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L^p(\mu), \ (1 < p \leq 2)$$

for any generator A of positive contraction C_0-semigroups in $L^p(\mu)$ with $1 < p \leq 2$. We show also that this last result is true in L^1 spaces or more generally (with an appropriate formulation in terms of canonical half norm N) in general *base-norm spaces* X. Actually, we show the much more general statement

$$N(x - Cx) \leq \|x\| \quad \forall x \in X \tag{4}$$

for any linear positive contraction C on a base-norm ordered Banach space X (see Theorem 8). An important feature of such spaces is the *additivity* of the norm on the positive cone which plays a key role in the proof of (4). In particular, for any linear positive contraction C in preduals $\mathcal{X}_*$ of hermitian part of von Neumann algebras $\mathcal{X}$ we have

$$\left\|(x - Cx)^{\pm}\right\|_{\mathcal{X}_*} \leq \|x\|_{\mathcal{X}_*} \quad \forall x \in \mathcal{X}_*$$

(see Corollary 11) where $\eta^{\pm} \geqslant 0$ refer to the unique Jordan decomposition of $\eta \in \mathcal{X}^*$, i.e. $\eta = \eta^+ - \eta^-$ with $\|\eta\|_{\mathcal{X}^*} = \|\eta^+\|_{\mathcal{X}^*} + \|\eta^-\|_{\mathcal{X}^*}$ $\forall \eta \in \mathcal{X}^*$ [10]. Such results imply e.g. "contractivity" properies of $I - \mathcal{E}$ for conditional expectations $\mathcal{E}$ in both contexts of classical and noncommutative probability (see Remark 32).

We provide an extension of M. Pierre's result above to general real ordered Banach spaces under a suitable monotony assumption on the duality map. More precisely, let

$$\zeta : \mathbb{R}_+ \to \mathbb{R}_+$$

be continuous, strictly increasing with $\zeta(0) = 0$, $\zeta(r) \to +\infty$ as $r \to +\infty$ and let

$$\Phi : X \to X'$$

be a duality map relative to the gauge ζ, i.e. such that

$$\langle \Phi(x), x \rangle = \zeta(\|x\|) \|x\| \quad \forall x \in X,$$

(such a duality map always exists). We show then that $A(\lambda - A)^{-1}$ is a contraction on the positive cone X_+, i.e.

$$\left\| A(\lambda - A)^{-1} x \right\| \leq \|x\| \quad \forall x \in X_+, \tag{5}$$

for any generator A of positive contraction C_0-semigroups provided the duality map Φ is uniformly monotone on the positive cone X_+ in the sense

$$\langle \Phi(x) - \Phi(y), x - y \rangle \geqslant \zeta(\|x - y\|) \|x - y\| \quad \forall x, y \in X_+, \tag{6}$$

(see Theorem 18). This inequality which is true in Lebesgue spaces $L^p(\mu)$ with $p \geqslant 2$ (see Theorem 12) admits a (non trivial) generalization to non-commutative L^p spaces by E. Ricard [20].

The mathematical results of this paper seem to single out the class $\mathcal{C}$ of ordered Banach spaces for which (5) is satisfied for *all* generators A of positive contraction C_0-semigroups (and we wonder if this class is much larger than the class of ordered Banach spaces whose duality map satisfies (6)). Indeed, for any generator A of a positive contraction C_0-semigroup on $X \in \mathcal{C}$, the relative A-bound of positive A-bounded operators S is given by

$$\lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)}$$

as in Hilbert space settings (see Theorem 21). For any generator A of a positive ergodic contraction C_0-semigroup with ergodic projection P on an ordered Banach space $X \in \mathcal{C}$

$$\|x - Px\| \leq \|x\| \ \forall x \in X_+,$$

i.e. $I - P$ is a contraction on the positive cone (see Theorem 22). Surprisingly enough, Theorem 22 implies the more general statement that

$$\|x - Cx\| \leq \|x\| \ \forall x \in X_+$$

for *any positive projection* C *with norm* 1 on an ordered Banach space $X \in \mathcal{C}$ (see Theorem 23). (Notice parenthetically that norm one projections in Banach spaces are intensively studied in the literature, see e.g. [11][19] and the references therein.)

I thank E. Ricard who kindly provided me with an elegant *direct* proof of this last result on ordered Banach spaces whose duality map satisfies (6) and

is single-valued on $X_+ - \{0\}$ (see Theorem 26). In particular, if $(\Omega, \mathcal{A}, P)$ is a probability space and if

$$\mathcal{E}^{\mathcal{B}} : f \in L^\infty(\Omega, \mathcal{A}, P) \to L^\infty(\Omega, \mathcal{B}, P)$$

is the conditional expectation with respect to a σ-subalgebra $\mathcal{B} \subset \mathcal{A}$ then

$$\left\| f - \mathcal{E}^{\mathcal{B}} f \right\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L_+^p(\Omega, \mathcal{A}, P) \quad (p > 2),$$

see Remark 33); (this result is also true in the noncommutative context, see the comments in the last section; I thank J. Ch. Bourin for a helpful discussion around this topic).

I am grateful to Hocine Mokhtar-Kharroubi for helpful informations on convex analysis; in particular, we "understand" why M. Pierre's result is not true in L^p spaces once $p < 2$ because of the fact that the map

$$X \ni x \to \|x\|^p$$

is *not* uniformly convex on the whole space for $p < 2$ (see Remark 20).

Another result observed by M. Pierre (private communication) is that, for any linear positive contraction C on $L^\infty(\mu)$, $I - C$ is a contraction on the positive cone $L_+^\infty(\mu)$. We extend this result to general *order-unit spaces* (see Theorem 29) and this extension, in turn, provides us with an alternative "shorter" proof (by a duality argument) of the contractivity property (4) in base-norm spaces (see Corollary 30).

The author is grateful to M. Pierre and E. Ricard for their help. This work owes very much to inspiring exchanges with both of them.

2 On relative operator bounds

Let $(U(t))_{t \geqslant 0}$ be a C_0-semigroup on a Banach space X with generator

$$A : D(A) \subset X \to X$$

and let

$$S : D(A) \to X$$

be bounded on $D(A)$ endowed with the graph norm, i.e. S is A-bounded. Let

$$s(A) := \sup \{\operatorname{Re} \lambda ; \lambda \in \sigma(A)\}$$

be the spectral bound of A. We start with a general observation.

Lemma 1 *Let a_S be the A-bound of S. Then*

$$\begin{aligned} a_S &\leq \inf\left\{\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)};\ \lambda > s(A)\right\} \\ &\leq \lim\sup_{\lambda\to+\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)} \\ &\leq a_S \lim\sup_{\lambda\to+\infty}\left\|A(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)}. \end{aligned}$$

In particular, if

$$\lim\sup_{\lambda\to+\infty}\left\|A(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)} \leq 1 \tag{7}$$

then

$$a_S = \lim_{\lambda\to+\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)}.$$

Proof:

Let $a > 0$ and $b > 0$ be such that

$$\|Sx\| \leq a\|Ax\| + b\|x\|,\ x \in D(A).$$

Let ω' be the type of $(U(t))_{t\geqslant 0}$ and let $\omega > \omega'$. There exists $M_\omega \geqslant 1$ such that $\|U(t)\| \leq M_\omega e^{\omega t}$ $\forall t \geqslant 0$. Then for $\lambda > \omega$

$$\begin{aligned} \left\|S(\lambda-A)^{-1}x\right\| &\leq a\left\|A(\lambda-A)^{-1}x\right\| + b\left\|(\lambda-A)^{-1}x\right\| \\ &\leq a\left\|A(\lambda-A)^{-1}x\right\| + \frac{bM_\omega}{\lambda-\omega}\|x\| \end{aligned}$$

and

$$\lim\sup_{\lambda\to+\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)} \leq a\lim\sup_{\lambda\to+\infty}\left\|A(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)}$$

so

$$\lim\sup_{\lambda\to+\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)} \leq a_S\lim\sup_{\lambda\to+\infty}\left\|A(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)}.$$

On the other hand, for any $\lambda > s(A)$,

$$\|Sx\| = \left\|S(\lambda-A)^{-1}(\lambda-A)x\right\| \leq \left\|S(\lambda-A)^{-1}\right\|\|Ax\| + \lambda\left\|S(\lambda-A)^{-1}\right\|\|x\|$$

shows that

$$a_S \leq \inf\left\{\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)};\ \lambda > s(A)\right\}$$

and this ends the proof. ■

Remark 2 *It follows from Lemma 1 that the A-bound of S is equal to zero if and only if*

$$\lim_{\lambda\to\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(X)}=0.$$

An important illustration of this result, in L^1 spaces, is provided by Kato class potentials and Schrödinger operators, see e.g. [23] Proposition A.2.3.

We have a more precise result in Hilbert spaces.

Corollary 3 *Let there exist some $\overline{\lambda}\in\mathbb{R}$ such that $A+\overline{\lambda}$ generates a contraction C_0-semigroup on a Hilbert space H and let $S:D(A)\to X$ be A-bounded. Then the A-bound of S is equal to*

$$\lim_{\lambda\to+\infty}\left\|S(\lambda-A)^{-1}\right\|_{\mathcal{L}(H)}.$$

Proof:
Let $A':=A+\overline{\lambda}$. We consider the equation

$$\lambda x-A'x=y\quad(\lambda>s(A)+\overline{\lambda}).$$

Then

$$\lambda(A'x,x)-\left\|A'x\right\|^2=(A'x,y)$$

and

$$-\lambda\operatorname{Re}(A'x,x)+\left\|A'x\right\|^2=-\operatorname{Re}(A'x,y)\leq\left\|A'x\right\|\left\|y\right\|.$$

The dissipativity of A', i.e. $\operatorname{Re}(A'x,x)\leq 0\ \forall x\in D(A)$ implies

$$\left\|A'x\right\|^2\leq\left\|A'x\right\|\left\|y\right\|$$

i.e.

$$\left\|A'(\lambda-A)^{-1}y\right\|\leq\left\|y\right\|$$

so that $A'(\lambda-A')^{-1}$ is a contraction. Thus

$$(A+\overline{\lambda})(\lambda-(A+\overline{\lambda}))^{-1}=A(\lambda-\overline{\lambda}-A)^{-1}+\overline{\lambda}(\lambda-\overline{\lambda}-A)^{-1}$$

is a contraction for all $\lambda>s(A)+\overline{\lambda}$ and then (7) holds since $\left\|\overline{\lambda}(\lambda-\overline{\lambda}-A)^{-1}\right\|$ goes to zero as $\lambda\to+\infty$. ■

Remark 4 *As pointed out to me by M. Pierre, the fact that $(1-\lambda^{-1}A)^{-1}-I$ is a contraction is known in the general context of nonlinear maximal dissipative operators A in Hilbert spaces (see [4] Proposition 2.6, p. 28).*

Remark 5 *In the general setting of Banach spaces, we note that if, for some* $\overline{\lambda} \in \mathbb{R}$, $A' := A + \overline{\lambda}$ *generates a contraction* C_0*-semigroup then*

$$\left\| A'(\lambda - A')^{-1} \right\|_{\mathcal{L}(X)} = \left\| -I + \lambda(\lambda - A')^{-1} \right\|_{\mathcal{L}(X)} \leq 2$$

which leads to the estimate

$$\lim \sup_{\lambda \to +\infty} \left\| A(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \leq 2$$

which cannot be improved in general, see (2).

3 Reminders on ordered Banach spaces

For reader's convenience, this section is devoted to reminders on some basic notions and results on real ordered Banach spaces we need thereafter. A real ordered Banach space is a triple $(X, X_+, \|\|)$ where $(X, \|\|)$ is a real Banach space with norm $\|\|$ and $X_+ \subset X$ is a closed convex cone. It follows that $(X', X'_+, \|\|')$ is also an ordered Banach space with the dual (weak*) closed convex cone

$$X'_+ := \left\{ x' \in X',\ \langle x', x \rangle \geqslant 0\ \forall x \in X_+ \right\}.$$

If X_+ is proper, i.e.

$$X_+ \cap -X_+ = \{0\},$$

then X_+ induces an order relation $\leq$ on X by

$$x \leq y \Leftrightarrow y - x \in X_+.$$

We say that X_+ is weakly generating if $X_+ - X_+$ is dense in X; in this case the dual cone X'_+ is proper. If

$$X = X_+ - X_+$$

then we say that X_+ is generating. If additionaly X_+ is normal, i.e. there exists $\alpha \geqslant 1$ such that $x \leq y \leq z$ implies $\|y\| \leq \alpha \{\|x\| \vee \|z\|\}$, then the dual cone X'_+ is also generating and normal. The norm is said to be absolutely monotone if

$$-y \leq x \leq y \Rightarrow \|x\| \leq \|y\|,$$

and approximately absolutely dominating if for any $\alpha > 1$

$$\forall x \in X,\ \exists y \in X_+,\ -y \leq x \leq y,\ \|y\| \leq \alpha \|x\|.$$

A norm is called a *Riesz norm* if it is both absolutely monotone and approximately absolutely dominating. Then $\|\|$ is a Riesz norm if and only if the dual norm is a Riesz norm. We mention a useful property of a Riesz norm $\|\|$

$$\|x\| = \inf \{\|y+z\| \,;\ x = y-z,\ y,z \in X_+\}\ \forall x \in X. \tag{8}$$

Note that a real ordered Banach space $(X, X_+, \|\|)$ is a *Banach lattice* if X is lattice (each pair $x, y \in X$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$) and $\|\|$ is a Riesz norm. The *canonical half-norm* on an ordered Banach space X is defined by

$$N(x) = dist(-x, X_+) = \inf \{\|x+y\|\,,\ y \in X_+\}$$

or equivalently by

$$N(x) = \inf \{\|z\|\,,\ z \in X,\ z \geqslant x\}\,. \tag{9}$$

In Banach lattices or on the self-adjoint part of a C^* algebra $N(x) = \|x_+\|$ where x_+ is the positive component of $x \in X$. For all these results (and many others), we refer to the exhaustive survey [2].

An ordered Banach space X is called an *order-unit space* if $IntX_+ \neq \emptyset$ and there exists $e \in IntX_+$ such that

$$\|x\| = \inf \{\lambda > 0;\ -\lambda e \leq x \leq \lambda e\}\,;$$

(we note that $X = \cup_{\lambda \geqslant 0} [-\lambda e, \lambda e]$ for any point $e \in IntX_+$). Order-unit spaces cover real AM-spaces (e.g. $L^\infty(\mu)$ or $C(K)$ spaces) or more generally hermitian part of C^* algebras.

Let $(X, X_+, \|\|)$ be a real ordered Banach space with a generating cone X_+. A *base* for X_+ is a bounded closed subset K of X_+ such that for each $x \in X_+$ there is a unique $\lambda_K(x) \geq 0$ such that $x \in \lambda_K(x)K$. In this case,

$$\|x\|_K := \inf \{\lambda \geq 0;\ x \in \lambda co(K \cup -K)\}$$

is a *Riesz norm* equivalent to the original norm $\|\|$. A real ordered Banach space $(X, X_+, \|\|)$ is called a *base-norm space* if there exists a base K for X_+ such that

$$\|x\|_K = \|x\|\,;$$

in such a case, the norm $\|.\|$ is *additive* on the positive cone, i.e.

$$\|x+y\| = \|x\| + \|y\| \quad \forall x, y \in X_+.$$

Finally, an ordered Banach space $(X, X_+, \|\|)$ is a base-norm space if and only if the dual ordered Banach space $(X', X'_+, \|\|')$ is an order-unit space. Similarly, $(X, X_+, \|\|)$ is an order-unit space if and only if $(X', X'_+, \|\|')$ is base-norm space. The typical examples of base-norm spaces are provided by AL spaces (e.g. $L^1(\mu)$ spaces or the space of Borel measures on a metric space endowed with the total variation norm), the duals of the hermitian part of C^* algebras or by the preduals of the hermitian part of von Neumann algebras. We refer to [2][15] for the details.

4 Contractivity theorems in ordered Banach spaces

To motivate what follows, let $(X, X_+, \|\|)$ be an ordered Banach space with a *Riesz norm* and let

$$A : D(A) \subset X \to X$$

be the generator of a positive (i.e. leaving invariant the positive cone X_+) C_0-semigroup $(U(t))_{t \geqslant 0}$ on X. Let

$$S : D(A) \to X$$

be positive, i.e.

$$S : D(A) \cap X_+ \to X_+,$$

and A-bounded. Since $S(\lambda - A)^{-1}$ is a positive operator then

$$\left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} = \sup_{\|x\| \leq 1,\ x \in X_+} \left\| S(\lambda - A)^{-1} x \right\|$$

because $\|\|$ is a Riesz norm (see [21] Lemma 3.3). Thus, if we resume the proof of Lemma 1 by using positive vectors $x \in X_+$ we end up with

$$a_S \leq \lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \leq a_S \lim \sup_{\lambda \to +\infty} \left(\sup_{\|x\| \leq 1,\ x \in X_+} \left\| A(\lambda - A)^{-1} x \right\| \right)$$

so

$$a_S = \lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)} \tag{10}$$

if

$$\lim \sup_{\lambda \to +\infty} \left(\sup_{\|x\| \leq 1,\ x \in X_+} \left\| A(\lambda - A)^{-1} x \right\| \right) \leq 1, \tag{11}$$

in particular if $A(\lambda - A)^{-1}$ is "a contraction on the positive cone" X_+ in the sense

$$\left\| A(\lambda - A)^{-1} x \right\| \leq \|x\| \quad \forall x \in X_+. \tag{12}$$

The contraction property on the positive cone admits a *dual* characterization.

Lemma 6 *Let $(X, X_+, \|\|)$ be an ordered Banach space and let $B \in \mathcal{L}(X)$. Then:*

1) B is a contraction on X_+ if and only if $N'(B'x') \leq \|x'\|_{X'} \quad \forall x' \in X'$.

2) $N(Bx) \leq \|x\|_X \quad \forall x \in X$ if and only if B' is a contraction on X'_+.

Proof:

(i) Let B be a contraction on X_+. Then for $x' \in X'$ and $x \in X_+$

$$\left|\langle B'x', x\rangle\right| = \left|\langle x', Bx\rangle\right| \leq \left\|x'\right\|_{X'} \|x\|_X .$$

Since

$$N'(y') = \sup_{x \in X_+,\ \|x\| \leq 1} \langle y', x\rangle, \quad \forall y' \in X'$$

(see [2] Proposition 1.6.2) then

$$N'\left(B'x'\right) = \sup_{x \in X_+,\ \|x\| \leq 1} \left|\langle B'x', x\rangle\right| \leq \left\|x'\right\|_{X'} .$$

(ii) Suppose now that $N(Bx) \leq \|x\|_X \quad \forall x \in X$. Since

$$N(y) = \sup_{x' \in X'_+,\ \|x'\| \leq 1} \langle x', y\rangle, \quad \forall y \in X$$

(see also [2] Proposition 1.6.2) then

$$\|x\|_X \geqslant N(Bx) = \sup_{x' \in X'_+,\ \|x'\| \leq 1} \langle x', Bx\rangle \geqslant \langle x', Bx\rangle \ \forall x' \in X'_+, \ \left\|x'\right\| \leq 1$$

so

$$\left|\langle B'x', x\rangle\right| = \left|\langle x', Bx\rangle\right| \leq \|x\|_X \left\|x'\right\|_{X'} \quad \forall x' \in X'_+, \forall x \in X$$

and then

$$\left\|B'x'\right\|_{X'} \leq \left\|x'\right\|_{X'} \quad \forall x' \in X'_+.$$

(iii) To show 1) we have just to show the converse part. Let $N'(B'x') \leq \|x'\|_{X'} \quad \forall x' \in X'$. By (ii), B'' (the bidual operator) is a contraction on the positive cone of X''. This ends the proof since B'' leaves invariant X and coincides with B on X.

(iv) To show 2) we have just to show the converse part. Let B' be a contraction on X'_+. Then (i) implies $N''(B''x'') \leq \|x''\|_{X''} \quad \forall x'' \in X''$. On the other hand (see [2] Proposition 1.6.2)

$$\begin{aligned} N''(z) &= \sup_{x' \in X'_+,\ \|x'\| \leq 1} \langle z, x'\rangle, \quad (\text{if } z \in X'') \\ &= N(z) \quad (\text{if } z \in X) \end{aligned}$$

so that $N(Bx) \leq \|x\|_X \quad \forall x \in X$ since B'' leaves invariant X and coincides with B on X. ■

Corollary 7 *Let $(U(t))_{t\geqslant 0}$ be a positive C_0-semigroup with generator A on an ordered Banach space $(X, X_+, \|\|)$.*

(i) $A(\lambda - A)^{-1}$ is a contraction on the positive cone X_+ if and only if

$$N'\left(A'(\lambda - A')^{-1}x'\right) \leq \left\|x'\right\|_{X'} \quad \forall x' \in X'.$$

(ii) $N\left(A(\lambda - A)^{-1}x\right) \leq \|x\| \quad \forall x \in X$ if and only if $A'(\lambda - A')^{-1}$ is a contraction on X'_+.

We start with a "contractivity" theorem in base-norm spaces. A key point is the *additivity* of the norm on the positive cone.

Theorem 8 *Let X be a base-norm ordered Banach space. If $C \in \mathcal{L}(X)$ is a positive contraction then*

$$N(x - Cx) \leq \|x\| \quad \forall x \in X.$$

Proof:

Let

$$y = x - Cx.$$

Let $x_1 \geqslant x$ and $z'_1 \geqslant -Cx$ be arbitrary. Then

$$\begin{aligned} y &= (x_1 - (x_1 - x)) + \left[z'_1 - \left(z'_1 + Cx\right)\right] \\ &= \left(x_1 + z'_1\right) - \left((x_1 - x) + \left(z'_1 + Cx\right)\right). \end{aligned}$$

By (9)

$$N(y) \leq \left\|x_1 + z'_1\right\| \leq \|x_1\| + \left\|z'_1\right\|.$$

The arbitrariness of x_1, z'_1 imply

$$N(y) \leq N(x) + N(-Cx).$$

On the other hand, since C is a positive contraction and because of the Riesz norm (see [21] Lemma 3.2) we have

$$N(Cz) \leq N(z) \quad \forall z \in X$$

whence

$$N(-Cx) \leq N(-x)$$

and then

$$N(y) \leq N(x) + N(-x).$$

On the other hand, the additivity of the norm on the positive cone implies

$$N(z) + N(-z) \leq \|z\| \;\; \forall z \in X.$$

Indeed, for any decomposition

$$z = z_1 - z_2, \;\; z_i \in X_+,$$

we have (by (9))

$$N(z) \leq \|z_1\|\,, \;\;\; N(-z) \leq \|z_2\|$$

so (using the additivity of the norm on X_+)

$$N(z) + N(-z) \leq \|z_1\| + \|z_2\| = \|z_1 + z_2\|$$

and then

$$N(z) + N(-z) \leq \|z_1 + z_2\|$$

which shows the claim thanks to the Riesz norm property (8). Finally $N(y) \leq \|x\| \;\;\; \forall x \in X.$ ■

Remark 9 *An alternative "shorter" proof of Theorem 8 is given in Corollary 30 as a consequence of a (different) "contractivity" result in order-unit spaces (Theorem 29).*

Theorem 8 and the fact that, for a positive contraction C_0-semigroup with generator A,

$$-A(\lambda - A)^{-1} = I - \lambda(\lambda - A)^{-1}$$

is a perturbation of the identity by a positive contraction yield:

Corollary 10 *Let X be a base-norm ordered Banach space. Let $(U(t))_{t\geqslant 0}$ be a positive contraction C_0-semigroup on X with generator A. Then*

$$N\left(A(\lambda - A)^{-1}y\right) \leq \|y\| \;\;\; \forall y \in X, \;\;\; \forall \lambda > 0$$

where N is the canonical half-norm in X.

A classical result by A. Grothendieck [10] asserts that if $\mathcal{X}$ is the hermitian part of a C^* algebra then each continuous linear functional η on $\mathcal{X}$ admits a unique (Jordan) decomposition

$$\eta = \eta_1 - \eta_2 \;\;\; (\eta_i \geqslant 0, \;\; i = 1, 2)$$

such that

$$\|\eta\|_{\mathcal{X}^*} = \|\eta_1\|_{\mathcal{X}^*} + \|\eta_2\|_{\mathcal{X}^*}$$

(see also [22] for more recent developments). Moreover ([22] Lemma 6)

$$N^*(\eta) = \|\eta_1\|_{\mathcal{X}^*}\,,\ N^*(-\eta) = \|\eta_2\|_{\mathcal{X}^*}\,.$$

On the other hand, if $\mathcal{X}$ is the hermitian part of a von Neumann algebra with predual $\mathcal{X}_*$ and if $\eta \in \mathcal{X}_*$ then $\eta_i \in \mathcal{X}_*$ $(i = 1, 2)$ [10]; thus

$$N_*(\eta) = \|\eta_1\|_{\mathcal{X}_*}\,,\ N_*(-\eta) = \|\eta_2\|_{\mathcal{X}_*}\,,\quad (\eta \in \mathcal{X}_*)$$

where N_* is the canonical norm on $\mathcal{X}_*$. By using the notations

$$\eta_+ := \eta_1,\quad \eta_- := \eta_2,$$

Theorem 8 yields:

Corollary 11 *Let $C \in \mathcal{L}(\mathcal{X}_*)$ be a positive contraction where $\mathcal{X}_*$ is the predual of (the hermitian part of) a von Neumann algebra $\mathcal{X}$. Then*

$$\left\|(x - Cx)^{\pm}\right\| \le \|x\| \quad \forall x \in \mathcal{X}_*.$$

We give now an important result by M. Pierre (private communication).

Theorem 12 *(M. Pierre) Let $(\Omega; \mu)$ be a measure space with a σ-finite measure μ and let $(U(t))_{t\geqslant 0}$ be a positive contraction C_0-semigroup on $L^p(\Omega;\mu)$ with generator A. Then, for any $\lambda > 0$, $A(\lambda - A)^{-1}$ is a contraction on the positive cone, i.e.*

$$\left\|A(\lambda - A)^{-1} f\right\|_{L^p} \le \|f\|_{L^p} \quad \forall f \in L^p_+(\Omega;\mu),\ \forall \lambda > 0, \tag{13}$$

provided that $2 \le p < \infty$.

Proof:

We recall that the positivity of a contraction semigroup $(U(t))_{t\geqslant 0}$ in $L^p(\Omega;\mu)$ is characterized by the dispersivity of its generator i.e.

$$\int_\Omega (Au)(u_+)^{p-1}\mu(dx) \le 0, \quad \forall u \in D(A)$$

when $1 < p < \infty$ (see e.g. [16] Theorem 1.2, p. 249). We can write (13) in the form

$$\left\|(I + \alpha\widetilde{A})^{-1} f - f\right\|_{L^p} \le \|f\|_{L^p} \quad \forall f \in L^p_+(\Omega;\mu)\ \forall \alpha > 0 \tag{14}$$

where $\widetilde{A} = -A$ and $\alpha = \lambda^{-1}$. Let

$$u = (I + \alpha\widetilde{A})^{-1} f;$$

then $u \geqslant 0$ since f is (and the resolvent $(I + \alpha\widetilde{A})^{-1}$ is positive). We have

$$u + \alpha\widetilde{A}u = f. \tag{15}$$

Let

$$q = p - 1.$$

By multiplying (15) by $u^q - f^q$ we get

$$\int_\Omega (u-f)(u^q - f^q) + \alpha \int_\Omega u^q \widetilde{A}u \ = \alpha \int_\Omega f^q \widetilde{A}u \ = \int_\Omega (f-u) f^q$$

and then

$$\int_\Omega u^q \widetilde{A}u \ \geqslant 0$$

implies

$$\int_\Omega (u-f)(u^q - f^q) \leq \int_\Omega (f-u) f^q \leq \|f-u\|_{L^p} \|f^q\|_{L^{p'}}$$

where $p' = \frac{p}{p-1}$ i.e.

$$\int_\Omega (u-f)(u^q - f^q) \leq \int_\Omega (f-u) f^q \leq \|f-u\|_{L^p} \|f\|_{L^p}^{p-1}.$$

Now using the identity

$$(a-b)(a^q - b^q) \geqslant |a-b|^{q+1}, \quad \forall a \geqslant 0,\ b \geqslant 0 \tag{16}$$

for $q \geqslant 1$ (i.e. for $p \geqslant 2$) we get

$$\int_\Omega |u-f|^p \leq \|f-u\|_{L^p} \|f\|_{L^p}^{p-1}$$

and then $\|f-u\|_{L^p} \leq \|f\|_{L^p}$ i.e. (14) holds. We note that (16) can be proved by supposing e.g. $a > b$, dividing (16) by b^{q+1} and using the relation

$$\forall x \geqslant 1,\ (x-1)(x^q - 1) \geqslant (x-1)^{q+1};$$

the last relation follows from the fact that $g(x) := x^q - 1 - (x-1)^q$ is non-decreasing on $[1, \infty[$ and $g(1) = 0$. ∎

Remark 13 *We point out that Theorem 12 is not true in $L^p(\mu)$ spaces for $1 \leq p < 2$, see Ricard's counter-example given in Remark 27 below. We give also in Theorem 15 (and Remark 17) below, for p close to* 1*, general counter-examples for submarkovian C_0-semigroups (see also Remark 20 (ii)).*

Remark 14 *We note that if A is the generator of positive contraction C_0-semigroup on $L^p(\Omega;\mu)$ with $p \in [1,2)$ then*

$$\left\|(A(\lambda - A)^{-1}f)^{\pm}\right\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L^p(\Omega;\mu),\ \forall \lambda > 0.$$

Indeed, the case $p \in (1,2)$ follows from Theorem 12 and Corollary 7 while the case $p = 1$ is covered by Theorem 8. In $L^2(\Omega;\mu)$ we have a much stronger result since $A(\lambda - A)^{-1}$ is a contraction on the whole space (see the proof of Corollary 3).

Theorem 15 *Let $(\Omega;\mu)$ be a metric measure space and let $(U(t))_{t\geqslant 0}$ be a positive contraction C_0-semigroup on $L^1(\Omega;\mu)$ with generator A. We assume that A satisfies the following dual smoothing effects:*

$$(\lambda - A')^{-1} : L^{\infty}(\Omega;\mu) \to C_b(\Omega) \tag{17}$$

and, for any $\overline{x} \in \Omega$,

$$((\lambda - A')^{-1}f_{\varepsilon})(\overline{x}) \to ((\lambda - A')^{-1}1)(\overline{x}) \quad \text{as } \varepsilon \to 0 \tag{18}$$

where f_ε is equal to -1 on the ball $B(\overline{x},\varepsilon)$ and equal to 1 *outside this ball. Then $A(\lambda - A)^{-1}$ is not a contraction on $L^1_+(\Omega;\mu)$. If $(U_p(t))_{t\geqslant 0}$ is a submarkovian C_0-semigroup with generator A_p with the dual "smoothing effects" (17)(18) on L^∞ then, for p close to* 1*, $A_p(\lambda - A_p)^{-1}$ is not a contraction on $L^p_+(\Omega;\mu)$.*

Proof:

Note first that if $A(\lambda - A)^{-1}$ is a contraction on $L^1_+(\Omega;\mu)$ then $A(\lambda - A)^{-1}$ is a contraction on the *whole* space $L^1(\Omega;\mu)$ because of

$$\begin{aligned}
\left\|A(\lambda - A)^{-1}f\right\| &= \left\|A(\lambda - A)^{-1}(f_+ - f_-)\right\| \\
&\leq \left\|A(\lambda - A)^{-1}f_+\right\| + \left\|A(\lambda - A)^{-1}f_-\right\| \\
&\leq \|f_+\| + \|f_-\| = \|f\|.
\end{aligned}$$

Since the continuous function $(\lambda - A')^{-1}1$ cannot vanish identically, there exists some $\overline{x} \in \Omega$ such that

$$((\lambda - A')^{-1}1)(\overline{x}) > 0.$$

Consider the equation

$$\lambda u_\varepsilon - A' u_\varepsilon = f_\varepsilon$$

where

$$f_\varepsilon = \begin{cases} -1 \text{ on } & B(\overline{x}, \varepsilon) \\ 1 \text{ on } & B^c(\overline{x}, \varepsilon) \end{cases}$$

and $B(\overline{x}, \varepsilon)$ is the ball centered at $\overline{x}$ with radius ε. Then (17)(18) show that $A'u$ is continuous on $B(\overline{x}, \varepsilon)$ and, by 18),

$$(A'u_\varepsilon)(\overline{x}) = \lambda u_\varepsilon(\overline{x}) - f(\overline{x}) = \lambda u_\varepsilon(\overline{x}) + 1 > 1$$

for ε small enough. Thus

$$\|f_\varepsilon\|_{L^\infty} = 1 < (A'u_\varepsilon)(\overline{x}) \leq \left\|A'u_\varepsilon\right\|_{L^\infty} = \left\|A'(\lambda - A')^{-1} f_\varepsilon\right\|_{L^\infty}$$

and $A(\lambda - A)^{-1}$ is not a contraction on $L^1(\Omega; \mu)$.

To deal with the second claim, suppose that there exists a sequence $p_n > 1$ such that $p_n \to 1$ and $A_{p_n}(\lambda - A_{p_n})^{-1}$ is a contraction on $L_+^{p_n}(\Omega; \mu)$ for all n. Then

$$\left\|A_{p_n}(\lambda - A_{p_n})^{-1} f\right\|_{L^{p_n}} \leq \|f\|_{L^{p_n}} \quad \forall f \in L_+^1(\Omega; \mu) \cap L_+^\infty(\Omega; \mu) \;\; \forall n$$

and passing to the limit as $p_n \to 1$ we get

$$\left\|A_1(\lambda - A_1)^{-1} f\right\|_{L^1} \leq \|f\|_{L^1} \quad \forall f \in L_+^1(\Omega; \mu) \cap L_+^\infty(\Omega; \mu)$$

which is not true by the first part of the proof since $L_+^1(\Omega; \mu) \cap L_+^\infty(\Omega; \mu)$ is dense in $L_+^1(\Omega; \mu)$. Hence there exists some $\widehat{p} > 1$ such that $A_p(\lambda - A_p)^{-1}$ is not a contraction on $L_+^p(\Omega; \mu)$ for all $p \in [1, \widehat{p}]$. ■

Remark 16 *The smoothing conditions (17)(18) cannot be dropped a priori. Indeed, the multiplication operateur by a measurable fonction* $-\sigma(.)$

$$A_p : \varphi \in D(T_p) \to -\sigma\varphi \in L^p$$

(where $\sigma(.) \geqslant 0$*) with domain* $D(A_p) = \{\varphi \in L^p;\ \sigma\varphi \in L^p\}$ *generates a positive contraction* C_0*-semigroup*

$$e^{tA_p}\varphi = e^{-\sigma(.)t}\varphi$$

having the peculiarity that

$$0 \leq e^{tA_p}\varphi \leq \varphi \;\; \forall \varphi \in L_+^p$$

so

$$0 \leq \varphi - e^{tA_p}\varphi \leq \varphi$$

and therefore $I - e^{tA_p}$ is a contraction on L^p_+ for all $p \in [1, \infty]$. Similarly $A_p(\lambda - A_p)^{-1}$ is a contraction on L^p_+ for all $p \in [1, \infty]$ since

$$A_p(\lambda - A_p)^{-1}\varphi = \frac{-\sigma\varphi}{\lambda + \sigma\varphi}.$$

Remark 17 *The smoothing conditions (17)(18) are satisfied for instance by generators of convolution C_0-semigroups on $\mathbb{R}^n$*

$$U_p(t) : \varphi \in L^p(\mathbb{R}^n) \to \int_{\mathbb{R}^n} \varphi(x-y)\mu_t(dy) \in L^p(\mathbb{R}^n) \quad (1 \leq p \leq +\infty)$$

where μ_t are sub-probability measures such that

$$\widehat{\mu_t}(\zeta) = e^{-tF(\zeta)}$$

and $F(.)$ (the characteristic exponent) is a negative-definite function (see e.g. [12]), provided that μ_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$, e.g. if $e^{-tF(.)} \in L^1(\mathbb{R}^n)$ (for $t > 0$). This covers e.g. the Laplacian, fractional diffusions, relativistic Schrödinger operators etc.

We provide now an extension of Theorem 12 to general real ordered Banach spaces X under a suitable assumption on the duality map. Let

$$\zeta : \mathbb{R}_+ \to \mathbb{R}_+$$

be continuous, strictly increasing and such that $\zeta(0) = 0$ and $\zeta(r) \to +\infty$ as $r \to +\infty$. We say that

$$\Phi : X \to X'$$

is a duality map relative to the gauge ζ if

$$\langle \Phi(x), x \rangle = \|\Phi(x)\|_{X'} \|x\| \quad \forall x \in X$$

and

$$\|\Phi(x)\|_{X'} = \zeta(\|x\|) \;\; \forall x \in X.$$

We recall that a duality map relative to a gauge ζ always exists. Indeed, for a vector x on the unit sphere of X, we set

$$\Phi(x) = \zeta(1)x^*$$

where x^* is chosen (via Hahn-Banach theorem) in $\partial N(x)$ where $N(x) = \|x\|$, and

$$\Phi(\lambda x) = \zeta(\lambda) x^* \quad \lambda \in \mathbb{R}_+ \quad \|x\| = 1.$$

Thus

$$\langle \Phi(x), x \rangle = \zeta(\|x\|) \, \|x\| \;\; \forall x \in X. \tag{19}$$

We note that

$$\langle \Phi(x), Ax \rangle \leq 0 \;\; \forall x \in D(A)$$

because

$$\begin{aligned} \langle \Phi(x), Ax \rangle &= \langle \Phi(\|x\| \frac{x}{\|x\|}), Ax \rangle \\ &= \|x\| \, \zeta(\|x\|) \langle \left(\frac{x}{\|x\|} \right)^*, A \frac{x}{\|x\|} \rangle \leq 0 \end{aligned}$$

since $\left(\frac{x}{\|x\|} \right)^* \in \partial N(\frac{x}{\|x\|})$ and A is dissipative. Finally (see e.g. [13] Proposition 2.1, p. 175)

$$\langle \Phi(x) - \Phi(y), x - y \rangle \geqslant 0 \; \forall x, y \in X.$$

Theorem 18 *Let X be an ordered Banach space and let $(U(t))_{t \geqslant 0}$ be a positive contraction C_0-semigroup on X with generator A. Let there exist a duality map $\Phi : X \to X'$ (relative to some gauge $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$) such that*

$$\langle \Phi(x) - \Phi(y), x - y \rangle \geqslant \langle \Phi(x - y), x - y \rangle \quad \forall x, y \in X_+. \tag{20}$$

Then

$$\left\| A(\lambda - A)^{-1} x \right\| \leq \|x\| \quad \forall x \in X_+, \; \forall \lambda > 0. \tag{21}$$

Proof:

In view of (19), our assumption (20) is equivalent to

$$\langle \Phi(x) - \Phi(y), x - y \rangle \geqslant \|x - y\| \, \zeta(\|x - y\|) \;\; \forall x, y \in X_+. \tag{22}$$

Given $x \in X_+$ and $\lambda > 0$, let $y \in X_+$ be the solution to

$$\lambda y - Ay = x.$$

Then

$$\begin{aligned} \langle \Phi(\lambda y) - \Phi(x), \lambda y - x \rangle &= \langle \Phi(\lambda y) - \Phi(x), Ay \rangle \\ &= \langle \Phi(\lambda y), Ay \rangle - \langle \Phi(x), Ay \rangle \\ &= \lambda^{-1} \langle \Phi(\lambda y), A(\lambda y) \rangle - \langle \Phi(x), Ay \rangle \\ &\leq -\langle \Phi(x), Ay \rangle = -\langle \Phi(x), \lambda y - x \rangle \\ &\leq \|\Phi(x)\|_{X'} \, \|\lambda y - x\| = \zeta(\|x\|) \, \|\lambda y - x\| \end{aligned}$$

so

$$\zeta^{-1}\left[\frac{\langle \Phi(\lambda y)-\Phi(x),\lambda y-x\rangle}{\|\lambda y-x\|}\right]\leq \|x\|\,.$$

On the other hand, by (22)

$$\frac{\langle \Phi(\lambda y)-\Phi(x),\lambda y-x\rangle}{\|\lambda y-x\|}\geqslant \zeta(\|\lambda y-x\|)$$

or equivalently

$$\zeta^{-1}\left[\frac{\langle \Phi(\lambda y)-\Phi(x),\lambda y-x\rangle}{\|\lambda y-x\|}\right]\geqslant \|\lambda y-x\|$$

whence

$$\|\lambda y-x\|\leq \|x\|$$

i.e. $\|Ay\|\leq\|x\|$ or

$$\left\|A(\lambda-A)^{-1}x\right\|\leq\|x\|$$

and we are done. ■

Remark 19 *Note that in $L^p(\mu)$ $(1<p<\infty)$ with the gauge $\zeta(r)=r^{p-1}$ we have $\Phi(f)=|f|^{p-2}f$ and then* (20) *amounts to*

$$\int (f^{p-1}-g^{p-1})(f-g)\geqslant \|f-g\|^p\quad \forall f,g\in L^p_+(\mu) \tag{23}$$

which appears in the proof of M. Pierre's result (Theorem 12) and follows from (16) which holds for $q:=p-1\geqslant 1$. Its noncommutative version (see (32)) is also true [20].

Remark 20 *(i) A helpful discussion with Hocine Mokhtar-Kharroubi allowed to link* (22) *to a uniform convexity assumption on the positive cone of the convex function*

$$X_+\ni x\to \Psi(x):=p(\|x\|)$$

in the sense

$$\Psi(x)-\Psi(y)\geqslant \Phi(y)(x-y)+\frac{1}{2}\|x-y\|\,\zeta(\|x-y\|)\ \forall x,y\in X_+$$

where $p(r):=\int_0^r\zeta(s)ds$. Indeed, by reversing the role of $x\in X_+$ and $y\in X_+$ and adding we obtain (22)*. This assumption itself can be derived from the following stronger one*

$$\Psi(\lambda x+(1-\lambda)y)\leq \lambda\Psi(x)+(1-\lambda)\Psi(y)-\frac{\lambda(1-\lambda)}{2}\|x-y\|\,\zeta(\|x-y\|)$$

for all $x, y \in X_+,\ \lambda \in]0,1[$ (see e.g. [25]).

(ii) We note that with the gauge $\zeta(r) = r^{p-1}$ and $1 < p < 2$, Condition (22) *never holds. Indeed, according to ([25] Theorem 2) this would imply that*

$$X \ni x \to \|x\|^p$$

is uniformly convex (on the whole cone X_+) but this is not true for $p < 2$ (see [25] Theorem 5). This corroborates why Theorem 12 is not true once $p < 2$ (see Remark 27 and Theorem 15).

We denote by $\mathcal{C}$ the class of ordered Banach spaces X for which (5) is satisfied by *all* generators A of positive contraction semigroups on X. According to Theorem 18, this class contains the class of ordered Banach spaces whose duality map is uniformly monotone on the positive cone X_+ in the sense (20).

The question on relative operator bounds which motivated initially this paper has a general positive answer within the class $\mathcal{C}$.

Theorem 21 *Let $X \in \mathcal{C}$ be an ordered Banach space with a Riesz norm. Let A be the generator of a positive C_0-semigroup $(U(t))_{t \geqslant 0}$ such that, for some real α, $\|U(t)\| \leq e^{\alpha t}$ $\forall t \geqslant 0$ and let $S : D(A) \to X$ be positive (i.e. $S : D(A) \cap X_+ \to X_+$) and A-bounded. Then the relative A-bound of S is equal to*

$$\lim_{\lambda \to \infty} \left\| S(\lambda - A)^{-1} \right\|_{\mathcal{L}(X)}.$$

Proof:

By assumption $\widehat{A} := A - \alpha$ generates a positive contraction semigroup. By assumption (or a consequence of Theorem 18) $\widehat{A}(\lambda - \widehat{A})^{-1}$ is a contraction on X_+ $\forall \lambda > 0$. Finally, for $\lambda > \alpha$,

$$A(\lambda - A)^{-1} = \widehat{A}\left[(\lambda - \alpha) - \widehat{A}\right]^{-1} + \alpha(\lambda - A)^{-1}$$

yields

$$\lim_{\lambda \to +\infty} \sup \Big(\sup_{\|x\| \leq 1,\ x \in X_+} \left\| A(\lambda - A)^{-1} x \right\| \Big) \leq 1$$

i.e. (11) is satisfied and consequently (10) holds. ■

Furthermore, the property (5) characterizing the class $\mathcal{C}$ has consequences on ergodic semigroups on $X \in \mathcal{C}$.

Theorem 22 *Let X be an ordered Banach space and let $(U(t))_{t\geqslant 0}$ be a positive contraction C_0-semigroup on X with a generator A satisfying (21) (e.g. let (20) be satisfied). If $(U(t))_{t\geqslant 0}$ is mean ergodic (e.g. if X is reflexive) with ergodic projection P then $I-P$ is a contraction on the positive cone, i.e.*

$$\|x-Px\|\leq\|x\| \ \ \forall x\in X_{+.}$$

Proof:
We recall that the mean ergodicity of $(U(t))_{t\geqslant 0}$ means that

$$Px:=\lim_{t\to+\infty}\frac{1}{t}\int_0^t U(t)xdt$$

exists for all $x\in X$. In this case, P is a projection on the kernel of A along the closure of the range of A. On the other hand, the mean ergodicity of $(U(t))_{t\geqslant 0}$ is equivalent to the existence of the strong limit

$$\lim_{\lambda\to 0_+}\lambda(\lambda-A)^{-1}x \ \ \forall x\in X$$

and we have

$$Px=\lim_{\lambda\to 0_+}\lambda(\lambda-A)^{-1}x \ \ \forall x\in X;$$

(see [6], Theorem 5.1, p. 123). By assumption (or according to Theorem 18),

$$\left\|A(\lambda-A)^{-1}x\right\|\leq\|x\| \quad \forall x\in X_+,\ \forall\lambda>0.$$

It suffices to note that

$$\left\|A(\lambda-A)^{-1}x\right\|=\left\|x-\lambda(\lambda-A)^{-1}x\right\|$$

and to pass to the limit as $\lambda\to 0$. ■

More generally, the positive projections with norm 1 on $X\in\ \mathcal{C}$ enjoy a remarkable property.

Theorem 23 *Let $X\in\ \mathcal{C}$ be an ordered Banach space. Then*

$$\|x-Cx\|\leq\|x\| \ \ \forall x\in X_+$$

for any positive projection C on X with norm 1.

Proof:

We define
$$U(t) : X \to X \ (t \in \mathbb{R})$$
by
$$U(t)x = e^{-t}x + (1 - e^{-t})Cx. \tag{24}$$
It is elementary to check that $(U(t))_{t\in\mathbb{R}}$ is a continuous group which is positive and contractive for $t \geq 0$. On the other hand
$$U(t) \to C \text{ as } t \to +\infty$$
(in operator norm) and C is its ergodic projection. Finally Theorem 22 ends the proof. ■

Remark 24 *Theorem 23 applies for instance in commutative or non-commutative L^p spaces with $p \geq 2$ since (20) is satisfied, (see Remark 19).*

Remark 25 *The semigroup (24) appears e.g. in [3] where C is a conditional expectation in $L^2(\mu)$ where μ is a probability measure.*

We give now a very nice *direct* proof of Theorem 23 (under slightly stronger assumptions) kindly communicated to the author by E. Ricard.

Theorem 26 *(E. Ricard) Let X be an ordered Banach space such that the duality map satisfies (20). If the norm $\|\|$ is smooth (i.e. Gâteaux differentiable) on $X_+ - \{0\}$ then*
$$\|x - Cx\| \leq \|x\| \ \ \forall x \in X_+$$
for any positive projection C on X with norm 1.

Proof:

Let $x \in X$. We consider the convex function
$$h : [0, 1] \ni t \to \|Cx + t(x - Cx)\| .$$
Since
$$\|Cx\| = \|C(Cx + t(x - Cx))\| \leq \|Cx + t(x - Cx)\|$$
then h reaches its minimum at $t = 0$ and then
$$\lim_{t\to 0_+} \frac{\|Cx + t(x - Cx)\| - \|Cx\|}{t} \geq 0.$$

Let $B_{X'}$ be the unit sphere of X' and let $J : X - \{0\} \to 2^{B_{X'}}$ be the duality map

$$J(x) = \left\{x' \in X';\ \left\|x'\right\| = 1,\ \langle x', x\rangle = \|x\|\right\}.$$

Let now $x \in X_+$ with $Cx \neq 0$ and $x - Cx \neq 0$. Since $\|\|$ is smooth at Cx then $J(Cx)$ is a singleton and

$$\lim_{t\to 0_+} \frac{\|Cx + t(x - Cx)\| - \|Cx\|}{t} = \langle J(Cx), x - Cx\rangle$$

(see e.g. [8] Corollary 1.5 (i), p. 5). Since $J(Cx)$ is a singleton then

$$\frac{\Phi(Cx)}{\zeta(\|Cx\|)} = J(Cx)$$

so

$$\langle \Phi(Cx), x - Cx\rangle \geq 0$$

or

$$\langle \Phi(x) - \Phi(Cx), x - Cx\rangle \leq \langle \Phi(x), x - Cx\rangle.$$

Now (20) (or equivalently (22)) implies

$$\begin{aligned}
\|x - Cx\|\,\zeta(\|x - Cx\|) &\leq \langle \Phi(x) - \Phi(Cx), x - Cx\rangle \\
&\leq \langle \Phi(x), x - Cx\rangle \\
&\leq \|\Phi(x)\|_{X'}\,\|x - Cx\| \\
&= \zeta(\|x\|)\,\|x - Cx\|
\end{aligned}$$

whence

$$\zeta(\|x - Cx\|) \leq \zeta(\|x\|)$$

and finally

$$\|x - Cx\| \leq \|x\|\ \ \forall x \in X_+$$

since $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing. ■

Remark 27 *(E. Ricard). Theorem 12 is not true in $L^p(\mu)$ for $1 \leq p < 2$. Indeed, let $s \in (0,1)$ and let $\Omega = \{0,1\}$ be endowed with a probability measure P_s where $P_s(\{0\}) = s$ and $P_s(\{1\}) = 1 - s$. Let C be the positive contractive projection (on the constants)*

$$Cf = \int_\Omega f(\omega)dP_s(\omega)$$

and consider the positive ergodic semigroup $(U(t))_{t\geqslant 0}$ *on* $L^p(\Omega, P_s)$ *where*

$$U(t)f = e^{-t}f + (1-e^{-t})Cf.$$

If (13) were true for any generator of positive contraction semigroup then Theorem 22 would imply that

$$\|f - Cf\| \le \|f\| \;\; \forall f \ge 0.$$

In particular, the choice $f(0) = \frac{1}{s}$ *and* $f(1) = 0$ *would imply*

$$(1-s)^p + (1-s)s^{p-1} \le 1$$

or

$$1 + s^{p-1}(1 - ps^{2-p} + s^{2-p}\varepsilon(s) - s) \le 1$$

(where $\varepsilon(s) \to 0$ *as* $s \to 0$*) which is violated for small positive* s *if* $1 \le p < 2$.

Remark 28 *Let* X *be (say) a reflexive ordered Banach space whose dual norm satisfies (20) and let* N *be its canonical half-norm. The following statements follow directly from the results above and a duality argument (Corollary 7 and Lemma 6).*

(i) If $(U(t))_{t\geqslant 0}$ *is a positive contraction* C_0*-semigroup on* X *with generator* A*. Then*

$$N\left(A(\lambda - A)^{-1}x\right) \le \|x\| \quad \forall x \in X.$$

(ii) If C *is a positive projection with norm* 1 *then*

$$N\left(x - Cx\right) \le \|x\| \quad \forall x \in X.$$

M. Pierre pointed out to me that, for any positive linear contraction C on $L^\infty(\mu)$, $I - C$ is a contraction on the positive cone $L^\infty_+(\mu)$. This result extends to general order-unit spaces.

Theorem 29 *Let* C *be a linear positive contraction on an order-unit space* X*. Then* $I - C$ *is a contraction on the positive cone* X_+.

Proof:

We recall again that an ordered Banach space X is called an order-unit space if $IntX_+ \neq \emptyset$ and there exists $e \in IntX_+$ such that

$$\|x\| = \inf\left\{\lambda > 0;\ -\lambda e \le x \le \lambda e\right\}. \tag{25}$$

Note that (25) implies that $\|e\| = 1$ and

$$-\|x\|\, e \leq x \leq \|x\|\, e \quad \forall x \in X. \tag{26}$$

Thus, for $x \in X_+$,

$$0 \leq Cx \leq \|x\|\, Ce$$

and

$$-x \leq Cx - x \leq \|x\|\, Ce - x$$

so that

$$-\|x\|\, e \ \leq Cx - x \leq \|x\|\, Ce.$$

On the other hand, by (26)

$$Ce \leq \|Ce\|\, e \leq e.$$

Hence

$$-\|x\|\, e \quad \leq Cx - x \leq \|x\|\, e$$

and then (25) implies

$$\|Cx - x\| \leq \|x\| \quad \forall x \in X_+$$

and ends the proof. ■

It is interesting to observe that this result, in turn, provides us with an alternative "shorter" proof of Theorem 8.

Corollary 30 *Let X be a base-norm ordered Banach space. If $C \in \mathcal{L}(X)$ is a positive contraction then*

$$N\,(x - Cx) \leq \|x\| \quad \forall x \in X.$$

Proof:

We know that X' is an order-unit space [15]. Let C' be the dual positive contraction on X'. It follows from Theorem 29 that $I - C'$ is a contraction on the positive cone X'_+ and consequently a duality argument (Lemma 6) ends the proof. ■

Remark 31 *Similarly, we could derive Theorem 29 from Theorem 8 by a duality argument (Lemma 6) since the dual of an order-unit space is a base-norm space [15].*

5 On conditional expectations

We end this paper with different remarks on "contractivity" properties of $I - \mathcal{E}$ for classical or non-commutative conditional expectations $\mathcal{E}$.

Remark 32 *The following statements are direct consequences of Theorem 8, Corollary 11 and Theorem 29.*

(1) Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$\mathcal{E}^{\mathcal{B}} : f \in L^{\infty}(\Omega, \mathcal{A}, P) \to L^{\infty}(\Omega, \mathcal{B}, P)$$

be the conditional expectation with respect to a σ-subalgebra $\mathcal{B} \subset \mathcal{A}$. Then

$$\left\| f - \mathcal{E}^{\mathcal{B}} f \right\|_{L^{\infty}} \leq \|f\|_{L^{\infty}} \quad \forall f \in L^{\infty}_{+}(\Omega, \mathcal{A}, P) \tag{27}$$

and

$$\left\| (f - \mathcal{E}^{\mathcal{B}} f)^{\pm} \right\|_{L^{1}} \leq \|f\|_{L^{1}} \quad \forall f \in L^{1}(\Omega, \mathcal{A}, P). \tag{28}$$

(2) Let $(\mathcal{X}, \varphi)$ be a quantum probability space (i.e. $\mathcal{X}$ is a von Neumann algebra and φ is a σ-weakly continuous state on $\mathcal{X}$), let $\mathcal{X}_0 \subset \mathcal{X}$ be a sub-von Neumann algebra and let

$$\mathcal{E}^{\mathcal{X}_0} : \mathcal{X} \to \mathcal{X}_0$$

be a conditional expectation (see e.g. [9] or [18] for the different definitions). Then

$$\left\| a - \mathcal{E}^{\mathcal{X}_0} a \right\|_{\mathcal{X}} \leq \|a\|_{\mathcal{X}} \quad \forall a \in \mathcal{X}_{+}. \tag{29}$$

Moreover, if $\mathcal{E}^{\mathcal{X}_0}$ is σ-weakly continuous then the dual operator $(\mathcal{E}^{\mathcal{X}_0})'$ leaves invariant the predual space $\mathcal{X}_*$ and its restriction to $\mathcal{X}_*$ (the so-called quantum reduction [17]) is such that

$$\left\| \left(a - (\mathcal{E}^{\mathcal{X}_0})' a\right)^{\pm} \right\|_{\mathcal{X}_*} \leq \|a\|_{\mathcal{X}_*} \quad \forall a \in \mathcal{X}_*, \tag{30}$$

(where $\eta_{\pm}$ refer to the unique Jordan decomposition of hermitian element $\eta = \eta_{+} - \eta_{-}$ such that $\eta_{\pm} \geqslant 0$ and $\|\eta\|_{\mathcal{X}_*} = \left\|\eta_{+}\right\|_{\mathcal{X}_*} + \left\|\eta_{+}\right\|_{\mathcal{X}_*}$ [10]).

We don't know if (27)(29) *belong to the folklore on the subject. Their dual versions* (28)(30) *seem to appear here for the first time.*

Remark 33 *It follows from Theorem 23 (or Theorem 26) and Remark 24 that*

$$\left\| f - \mathcal{E}^{\mathcal{B}} f \right\|_{L^{p}} \leq \|f\|_{L^{p}} \quad \forall f \in L^{p}_{+}(\Omega, \mathcal{A}, P) \ \ (p > 2) \tag{31}$$

and then (because of the self-adjointness of $\mathcal{E}^{\mathcal{B}}$ on $L^2(\Omega, \mathcal{A}, P)$) Lemma 6 implies

$$\left\| (f - \mathcal{E}^{\mathcal{B}} f)^{\pm} \right\|_{L^{p}} \leq \|f\|_{L^{p}} \quad \forall f \in L^{p}(\Omega, \mathcal{A}, P) \quad (1 \leq p < 2).$$

We end this section with a comment on the non-commutative analogue of (31). Let H be a Hilbert space and let $\mathcal{X}$ be the *self-adjoint* part of a von Neumann algebra on H (i.e. a C^*-subalgebra of $B(H)$ which contains I and is w^*-closed). Let τ be a finite faithful trace normalized by $\tau(I) = 1$ and let $L^p(\mathcal{X}, \tau)$ be the completion of $\mathcal{X}$ endowed with the norm $\|\|_p$ where

$$\|x\|_p^p := \tau(|x|^p) \quad (p \geq 1)$$

where $|x| = \sqrt{x^2}$. Let $L^\infty(\mathcal{X}, \tau) = \mathcal{X}$ endowed with the operator norm on H. Then we can identify (isometrically) the dual of $L^p(\mathcal{X}, \tau)$ (with $1 \leq p < \infty$) to $L^{p^*}(\mathcal{X}, \tau)$ where p^* is the conjugate exponent and the duality pairing is given by

$$\langle x, y \rangle_{L^p, L^{p^*}} = \tau(xy).$$

Let $\mathcal{X}_0 \subset \mathcal{X}$ be a sub-von Neumann algebra. Then the conditional expectation

$$\mathcal{E}^{\mathcal{X}_0} : \mathcal{X} \to \mathcal{X}_0$$

extends to a positive contractive projection from $L^p(\mathcal{X}, \tau)$ to $L^p(\mathcal{X}_0, \tau)$ ($1 \leq p < \infty$) still denoted by $\mathcal{E}^{\mathcal{X}_0}$. We refer e.g. to [26] for the details.

Thus the non-commutative version of (23) (i.e. the condition (20) in $L_+^p(\mathcal{X}, \tau)$) amounts to

$$\tau\left[(x^{p-1} - y^{p-1})(x - y)\right] \geqslant \tau(|x - y|^p) \quad \forall x, y \in L_+^p(\mathcal{X}, \tau) \tag{32}$$

and has been proved by E. Ricard for $p \geq 2$ [20]. It follows from Theorem 23 that

$$\left\|x - \mathcal{E}^{\mathcal{X}_0} x\right\|_{L^p(\mathcal{X}, \tau)} \leq \|x\|_{L^p(\mathcal{X}, \tau)} \quad \forall x \in L_+^p(\mathcal{X}, \tau) \quad (p \geq 2);$$

see also [20].

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